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Radiation damping of the electron in a gravitational field

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Abstract. The radiation damping terms in the equation of motion of an electron in a curved space are deduced by the method of analytic continuation. This method works in curved space with essentially the same simplicity as in the flat space case. The contrast between the laborious usual procedure and the simplicity of the present one is remarkable.

1. Introduction

The radiation terms of the Lorentz–Dirac equation of motion for an electron have been recently obtained (Barut 1974, Barut and Villarroel 1975) by an extremely simple method starting from the equation

$$m\ddot{z}^{\alpha} = eF^{\mathrm{in}\alpha}{}_{\beta}\dot{z}^{\beta}. \tag{1.1}$$

The method consists of a kind of analytical prolongation of equation (1.1), when the retarded field $F_{\mu\nu}$ of the electron is also taken into account through the equation

$$m_0\ddot{z}^{\mu}(\tau+\delta\tau) = e[F^{\mathrm{in}\mu}{}_{\nu}(x = z(\tau+\delta\tau))+F^{\mu}{}_{\nu}(x = z(\tau+\delta\tau)), z(\tau)]\dot{z}^{\nu}(\tau+\delta\tau), \tag{1.2}$$

proceeding then to expand the second term of this equation in powers of $\delta\tau$. The divergent term that appears when $\delta\tau \to 0$ is absorbed, as usual, by a mass renormalization. The Lorentz–Dirac equation then follows by taking the limit $\delta\tau \to 0$. The purpose of this note is to obtain the equation of motion for the electron in a Riemannian space by the same method.

The equation of motion of an electron in a curved space has been studied by DeWitt and Brehme (1960) and Hobbs (1968) following essentially the procedure of Dirac (1938), that is, by computing the flux of the energy–momentum tensor across a thin tube that surrounds the electron world line. This method is rather laborious even in flat space. Recently (Tabensky and Villarroel 1975) the derivation of the Lorentz–Dirac equation has been simplified by studying the four-momentum of the electron's electromagnetic field. Unfortunately this concept is useless in a curved space. As we will show below, the method of analytic continuation works in curved space with essentially the same simplicity as it works in flat space.

In flat space the expansion of the second term of equation (1.2) is made in a Lorentz covariant way. In a curved space we must preserve, of course, covariance under arbitrary transformation of coordinates. A very useful quantity that allows covariant expansions

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is the characteristic function, or world function, $\sigma(x, z)$. If we designate by s the geodesic interval, which gives the magnitude of the invariant distance between x and z as measured along the geodesic joining them, the world function $\sigma(x, z)$ is defined by

$$\sigma = \pm\tfrac{1}{2}s^2 \tag{1.3}$$

which is positive for space-like intervals and negative for time-like ones. We will be using the same symbols, notation and conventions as DeWitt and Brehme (1960, to be hereafter referred to as DB). The signature of $g_{\mu\nu}$ is given by $(-+++)$. Covariant differentiation is denoted by a dot. Indices taken from the letters α to κ in the Greek alphabet are always to be associated with the point z, while indices taken from λ to ω are always to be associated with the point x. The electron world line is given by a set of functions $z^\alpha(\tau)$, where τ is the proper time. Dots over the z's denote absolute covariant differentiation with respect to τ. Thus

$$\begin{aligned} \dot{z}^\alpha &= \mathrm{d}z^\alpha/\mathrm{d}\tau \\ \ddot{z}^\alpha &= \mathrm{d}\dot{z}^\alpha/\mathrm{d}\tau + \Gamma_{\beta\gamma}{}^{\alpha}\dot{z}^\beta\dot{z}^\gamma \end{aligned} \tag{1.4}$$

etc. We also choose $c = 1$; therefore

$$\begin{aligned} \dot{z}^\alpha\dot{z}_\alpha &= -1 \\ \dot{z}^\alpha\ddot{z}_\alpha &= 0 \\ \dot{z}^\alpha\dddot{z}_\alpha &= -\ddot{z}^\alpha\ddot{z}_\alpha \equiv -\ddot{z}^2. \end{aligned} \tag{1.5}$$

The covariant retarded field of the electron is given by (DB, equation (3.52)).

$$\begin{aligned} F_{\mu\nu} = e\Big(&-\kappa^{-2}\sigma_{\cdot\mu\alpha}u_{\nu\beta}\dot{z}^\alpha\dot{z}^\beta + \chi\kappa^{-3}\sigma_{\cdot\mu}u_{\nu\alpha}\dot{z}^\alpha - \kappa^{-2}\sigma_{\cdot\mu}u_{\nu\alpha}(\ddot{z}^\alpha - \kappa'\kappa^{-1}\dot{z}^\alpha) - \kappa^{-1}u_{\nu\alpha\cdot\nu}\dot{z}^\alpha \\ &-\kappa^{-2}\sigma_{\cdot\mu}u_{\nu\alpha\cdot\beta}\dot{z}^\alpha\dot{z}^\beta + \kappa^{-1}\sigma_{\cdot\mu}v_{\nu\alpha}\dot{z}^\alpha + \int_{-\infty}^{\tau_-} v_{\mu\alpha\cdot\nu}\dot{z}^\alpha\,\mathrm{d}\tau - (\mu\leftrightarrow\nu)\Big). \end{aligned} \tag{1.6}$$

The field $F_{\mu\nu}$ is a bi-tensor that depends on the point x where the field is evaluated, and on the point z which is the retarded point associated to x, that is, it is defined by

$$\sigma(x, z(\tau_-)) = 0. \tag{1.7}$$

This is the world function that appears in equation (1.6). In addition, the other terms in this equation are defined as follows: $\kappa = \sigma_{\cdot\alpha}\dot{z}^\alpha$, $\kappa' = \sigma_{\cdot\alpha}\ddot{z}^\alpha$, $\chi = \sigma_{\cdot\alpha\beta}\dot{z}^\alpha\dot{z}^\beta$. The bi-vector $u_{\mu\alpha}$ is given by (DB, equation (2.50))

$$u_{\mu\alpha} = \Delta^{1/2}\bar{g}_{\mu\alpha} \tag{1.8}$$

where $\bar{g}_{\mu\alpha}(x, z)$ is the bi-vector of geodesic parallel displacement. The bi-scalar Δ is defined by (DB, equation (1.60))

$$\Delta = \bar{g}^{-1}D \tag{1.9}$$

where D and $\bar{g}$ are the following determinants:

$$\begin{aligned} D &= -|-\sigma_{\cdot\mu\alpha}| \\ \bar{g} &= -|\bar{g}_{\mu\alpha}|. \end{aligned} \tag{1.10}$$

The bivector $v_{\mu\alpha}$, the 'tail', is a complicated function of the metric. There is no known simple expression like (1.8) for it.

We emphasize that we use only the retarded field. The introduction of the advanced field used by DeWitt and Brehme (1960) and Hobbs (1968) is superfluous, even if we follow their procedure.

2. The radiation damping terms

In order to carry out the expansion of the second term of equation (1.2) in a covariant way, we use the same techniques as in DB. The world function $\sigma(x, z)$ associated with the points x and z is the structural element on which these covariant expansions are based. If we have a bi-tensor, say $T_{\alpha\beta}$, whose indices refer to the same point z, we can expand it about z in the covariant form

$$T_{\alpha\beta}(x, z) = A_{\alpha\beta} + A_{\alpha\beta}{}^{\gamma}\sigma_{\cdot\gamma} + \tfrac{1}{2}A_{\alpha\beta}{}^{\gamma\delta}\sigma_{\cdot\gamma}\sigma_{\cdot\delta} + \mathrm{O}(s^3) \tag{2.1}$$

where s is the length of the geodesic joining x and z. The coefficients $A_{\alpha\beta}$, etc are ordinary local tensors at z. They can be determined easily from the equations

$$\sigma_{\cdot\mu}\sigma^{\cdot\mu} = \sigma_{\cdot\alpha}\sigma^{\cdot\alpha} = 2\sigma. \tag{2.2}$$

Now, if we have a bi-tensor whose indices do not all refer to the point z, like the field $F_{\mu\nu}$ of equation (1.6), we define a new bi-tensor all of whose indices do refer to the point z with the help of the bi-vector of geodesic parallel displacement $\bar{g}_{\mu\alpha}$, and then we expand this new bi-tensor by means of equation (2.1).

As equation (1.2) shows, we need to evaluate the field $F_{\mu\nu}$ given by equation (1.6) at the point $x = z(\tau + \delta\tau)$. Therefore the point $z(\tau)$ is not the retarded one associated with x. The world function that will appear in our expansions is associated with the geodesic that joins the points $x = z(\tau + \delta\tau)$ and $z(\tau)$. But in the last instance our parameter in the expansion of equation (1.2) is $\delta\tau$. For this reason it is necessary to find the coefficients A_α, B_α, etc in

$$\sigma_{\cdot\alpha}(x(\tau + \delta\tau), z(\tau)) = A_\alpha\delta\tau + B_\alpha\delta\tau^2 + C_\alpha\delta\tau^3 + \mathrm{O}(\delta\tau^4) \tag{2.3}$$

where $A_\alpha, B_\alpha \ldots$ are local vectors at the point $z(\tau)$. For this purpose let us consider the quantity $\sigma_{\cdot\alpha}(x(\tau + \delta\tau), z(\tau^*))$. Of course we can write

$$\sigma_{\cdot\alpha}(x(\tau + \delta\tau), z(\tau^*)) = \sigma_{\cdot\alpha} + \dot{\sigma}_{\cdot\alpha}\delta\tau + \tfrac{1}{2}\ddot{\sigma}_{\cdot\alpha}\delta\tau^2 + \tfrac{1}{6}\dddot{\sigma}_{\cdot\alpha}\delta\tau^3 + \mathrm{O}(\delta\tau^*). \tag{2.4}$$

The geodesic of the world function on the left-hand side of this equation is the one joining the points $x(\tau + \delta\tau)$ and $z(\tau^*)$. But the corresponding geodesic on the right-hand side is the one joining the points $x(\tau)$ and $z(\tau^*)$. Now as the τ derivatives $\dot{\sigma}_\alpha$, $\ddot{\sigma}_\alpha$, etc are evaluated at the point $x(\tau)$, we can express them in the manifestly covariant forms (DB, equations (3.8), (3.9), etc)

$$\begin{aligned}
\dot{\sigma}_\alpha &= \sigma_{\cdot\alpha\mu}\dot{x}^\mu \\
\ddot{\sigma}_{\cdot\alpha} &= \sigma_{\cdot\alpha\mu\nu}\dot{x}^\mu\dot{x}^\nu + \sigma_{\cdot\alpha\mu}\ddot{x}^\mu \\
\dddot{\sigma}_{\cdot\alpha} &= \sigma_{\cdot\alpha\mu\nu\omega}\dot{x}^\mu\dot{x}^\nu\dot{x}^\omega + 3\sigma_{\cdot\alpha\mu\nu}\ddot{x}^\mu\dot{x}^\nu + \sigma_{\cdot\alpha\mu}\dddot{x}^\mu.
\end{aligned} \tag{2.5}$$

In order to find the coefficients in equation (2.3) we need the following results (DB, equations (1.18), (1.52), (1.68), (1.69)):

$$\begin{aligned}
&\lim_{x\to z} \sigma_{\cdot\alpha} = 0 \\
&\lim_{x\to z} \sigma_{\cdot\alpha\mu} = -g_{\mu\alpha} \\
&\lim_{x\to z} \sigma_{\cdot\alpha\mu\nu} = 0 \\
&\lim_{x\to z} \sigma_{\cdot\alpha\mu\nu\omega} = \lim_{x\to z} \bar{g}^{\tau}{}_{\alpha}(\tfrac{2}{3}R_{\tau\nu\mu\omega} - \tfrac{1}{3}R_{\tau\omega\mu\nu}).
\end{aligned} \tag{2.6}$$

Taking the limit $x(\tau) \to z(\tau^*)$ in equation (2.4) and using (2.5), (2.6) and the symmetries of the Riemann tensor, we find

$$\sigma_{\cdot\alpha}(x(\tau+\delta\tau), z(\tau)) = -\dot{z}_\alpha\delta\tau - \tfrac{1}{2}\ddot{z}_\alpha\delta\tau^2 - \tfrac{1}{6}\dddot{z}_\alpha\delta\tau^3 + \mathrm{O}(\delta\tau^4). \tag{2.7}$$

In particular, from this equation and equation (1.5) it follows that

$$\kappa = \sigma_{\cdot\alpha}(x(\tau+\delta\tau), z(\tau))\dot{z}^\alpha(\tau) = \delta\tau(1 + \tfrac{1}{6}\ddot{z}^2\delta\tau^2 + \mathrm{O}(\delta\tau^3)). \tag{2.8}$$

By a similar procedure, we find

$$\begin{aligned}
\kappa' &= \sigma_{\cdot\alpha}\ddot{z}^\alpha = -\tfrac{1}{2}\ddot{z}^2\delta\tau^2 + \mathrm{O}(\delta\tau^3) \\
\chi &= \sigma_{\cdot\alpha\beta}\dot{z}^\alpha\dot{z}^\beta = -1 + \mathrm{O}(\delta\tau^3).
\end{aligned} \tag{2.9}$$

With the help of (2.7), (2.8) and (2.9) it is easy to compute the second term of equation (1.2). Let us consider for example the expansion of the term

$$-\kappa^{-2}\sigma_{\cdot\mu\alpha}u_{\nu\beta}\dot{z}^\alpha\dot{z}^\beta \tag{2.10}$$

that appears in equation (1.6). Following the method described at the beginning of this section, we obtain the following results (see DB for details):

$$\sigma_{\cdot\mu\alpha} = -\bar{g}_{\mu\alpha} + \tfrac{1}{6}\bar{g}_\mu{}^\delta R^\delta{}_{\beta\alpha\gamma}\sigma^{\cdot\gamma}\sigma^{\cdot\beta} + \mathrm{O}(s^3) \tag{2.11}$$

and

$$u_{\nu\beta} = (1 - \tfrac{1}{12}R^{\beta\gamma}\sigma_{\cdot\beta}\sigma_{\cdot\gamma} + \mathrm{O}(s^3))\bar{g}_{\nu\beta}$$

where $R_{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta}$ are the Riemann and Ricci tensors respectively. Therefore,

$$-\kappa^{-2}\sigma_{\cdot\underline{\mu\alpha}}u_{\underline{\nu}\beta}\dot{z}^\alpha\dot{z}^\beta = \bar{g}_{\underline{\mu}\alpha}\bar{g}_{\underline{\nu}\beta}(-\tfrac{1}{6}R^\alpha{}_{\gamma\delta\epsilon}\sigma_{\cdot}{}^{\gamma}\sigma_{\cdot}{}^{\epsilon}\dot{z}^\beta\dot{z}^\delta + \mathrm{O}(s^3))\kappa^{-2} \tag{2.12}$$

where we have introduced the notation

$$A_{\dots\underline{\mu\dots\nu}\dots} = A_{\dots\mu\dots\nu\dots} - A_{\dots\nu\dots\mu\dots}\,. \tag{2.13}$$

Now, from equations (2.7) and (2.8) it follows that

$$-\kappa^{-2}\sigma_{\cdot\underline{\mu\alpha}}u_{\underline{\nu}\beta}\dot{z}^\alpha\dot{z}^\beta = \mathrm{O}(\delta\tau). \tag{2.14}$$

This shows that the term (2.10) does not contribute to equation (1.2) when $\delta\tau \to 0$.

Straightforward computation gives the following results for the other terms of equation (1.6):

$$
\begin{aligned}
\chi\kappa^{-3}\sigma_{\cdot\mu}u_{\nu\alpha}\dot{z}^{\alpha} &= \tfrac{1}{2}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}(\dot{z}^{\alpha}\dddot{z}^{\beta}\delta\tau^{-1}+\tfrac{1}{3}z^{\alpha}\ddddot{z}^{\beta})+\mathrm{O}(\delta\tau)\\
-\kappa^{-2}\sigma_{\cdot\mu}u_{\nu\alpha}(\dddot{z}^{\alpha}-\kappa'\kappa^{-1}\dot{z}^{\alpha}) &= -\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\dot{z}^{\alpha}\dddot{z}^{\beta}\delta\tau^{-1}+\mathrm{O}(\delta\tau)\\
-\kappa^{-1}u_{\mu\alpha\cdot\nu}\dot{z}^{\alpha} &= \bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}(\tfrac{1}{6}R^{\beta}{}_{\gamma}\dot{z}^{\alpha}\dot{z}^{\gamma}+\tfrac{1}{2}R_{\gamma}{}^{\alpha\beta}{}_{\delta}\dot{z}^{\gamma}\dot{z}^{\delta})+\mathrm{O}(\delta\tau)\\
-\kappa^{-2}\sigma_{\cdot\mu}u_{\nu\alpha\cdot\beta}\dot{z}^{\alpha}\dot{z}^{\beta} &= \mathrm{O}(\delta\tau)\\
\kappa^{-1}\sigma_{\cdot\mu}v_{\nu\alpha}\dot{z}^{\alpha} &= -\tfrac{1}{2}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}(R_{\gamma}{}^{\beta}-\tfrac{1}{6}\delta_{\gamma}{}^{\beta}R)\dot{z}^{\alpha}\dot{z}^{\gamma}+\mathrm{O}(\delta\tau)\\
\int_{-\infty}^{\tau} v_{\mu\alpha\cdot\nu}\dot{z}^{\alpha}\,\mathrm{d}\tau &= \bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\int_{-\infty}^{\tau} v^{\alpha}{}_{\gamma'}{}^{\beta}(z(\tau),z(\tau'))\dot{z}^{\gamma'}(\tau')\,\mathrm{d}\tau'+\mathrm{O}(\delta\tau).
\end{aligned}
\tag{2.15}
$$

Combining equations (2.14), (2.15) and (1.2) we obtain

$$
\begin{aligned}
m_0\ddot{z}_{\mu}(\tau+\delta\tau) &= eF^{\mathrm{in}}{}_{\mu\nu}(z(\tau+\delta\tau))\dot{z}^{\nu}(\tau+\delta\tau)+e^2(\bar{g}_{\mu\beta}\bar{g}_{\nu\gamma}-\bar{g}_{\nu\beta}\bar{g}_{\mu\gamma})\dot{z}^{\nu}(\tau+\delta\tau)\\
&\quad\times\Big(-\tfrac{1}{2}\dot{z}^{\beta}\dddot{z}^{\gamma}\delta\tau^{-1}+\tfrac{1}{6}\dot{z}^{\beta}\ddddot{z}^{\gamma}-\tfrac{1}{3}R^{\gamma}{}_{\delta}\dot{z}^{\beta}\dot{z}^{\delta}+\tfrac{1}{2}R_{\delta}{}^{\beta\gamma}{}_{\eta}\dot{z}^{\delta}\dot{z}^{\eta}\\
&\quad+\tfrac{1}{2}\int_{-\infty}^{\tau} f^{\beta\gamma}{}_{\delta'}(z(\tau),z(\tau'))\dot{z}^{\delta'}(\tau')\,\mathrm{d}\tau'+\mathrm{O}(\delta\tau)\Big)
\end{aligned}
\tag{2.16}
$$

where

$$f_{\mu\nu\alpha} = v_{\mu\alpha\cdot\nu}-v_{\nu\alpha\cdot\mu}. \tag{2.17}$$

By definition of parallel displacement we have that

$$\bar{g}_{\nu\beta}\dot{z}^{\nu}(\tau+\delta\tau) = \dot{z}_{\beta}+\ddot{z}_{\beta}\delta\tau+\mathrm{O}(\delta\tau^2) \tag{2.18}$$

where the coefficients on the right-hand side are evaluated at the proper time τ. In view of equation (2.18) and

$$\bar{g}^{\mu\alpha}\bar{g}_{\mu\beta} = \delta_{\beta}{}^{\alpha} \tag{2.19}$$

we can write (2.16) in the form

$$
\begin{aligned}
m_0\bar{g}^{\mu\alpha}\ddot{z}_{\mu}(\tau+\delta\tau) &= e\bar{g}^{\mu\alpha}F^{\mathrm{in}}{}_{\mu\nu}(z(\tau+\delta\tau))\dot{z}^{\nu}(\tau+\delta\tau)\\
&\quad+e^2\Big(-(1/2\delta\tau)\dddot{z}^{\alpha}-\tfrac{2}{3}\ddot{z}^2\dot{z}^{\alpha}+\tfrac{1}{6}\ddddot{z}^{\alpha}-\tfrac{1}{3}(R_{\beta\gamma}\dot{z}^{\alpha}\dot{z}^{\beta}\dot{z}^{\gamma}+R^{\alpha}{}_{\beta}\dot{z}^{\beta})\\
&\quad+\dot{z}_{\beta}\int_{-\infty}^{\tau} f^{\alpha\beta}{}_{\gamma'}\dot{z}^{\gamma'}\,\mathrm{d}\tau'+\mathrm{O}(\delta\tau)\Big)
\end{aligned}
\tag{2.20}
$$

by multiplying both terms by $\bar{g}^{\mu\alpha}$ and using equation (1.5). The Riemann tensor has disappeared because

$$R_{\delta\alpha\beta\eta}-R_{\delta\beta\alpha\eta} \tag{2.21}$$

is skew-symmetric in the indices δ, η.

In order to perform mass renormalization we rewrite equation (2.20) as follows:

$$
\begin{aligned}
m_0\bar{g}^{\mu\alpha}\ddot{z}_{\mu}(\tau+\delta\tau) &= e\bar{g}^{\mu\alpha}F^{\mathrm{in}}{}_{\mu\nu}(z(\tau+\delta\tau))\dot{z}^{\nu}(\tau+\delta\tau)-(e^2/2\delta\tau)\bar{g}^{\mu}{}_{\alpha}\dddot{z}_{\mu}(\tau+\delta\tau)+\tfrac{2}{3}e^2(\dddot{z}^{\alpha}-\ddot{z}^2\dot{z}^{\alpha})\\
&\quad-\tfrac{1}{3}e^2(R^{\alpha}{}_{\beta}\dot{z}^{\beta}+\dot{z}^{\alpha}R_{\beta\gamma}\dot{z}^{\beta}\dot{z}^{\gamma})+e^2\dot{z}^{\beta}\int_{-\infty}^{\tau} f^{\alpha}{}_{\beta\gamma'}\dot{z}^{\gamma'}\,\mathrm{d}\tau'+\mathrm{O}(\delta\tau)
\end{aligned}
\tag{2.22}
$$

where we have used the analogue of equation (2.18) for the vector $\bar{g}^{\mu\alpha}\ddot{z}_{\mu}(\tau+\delta\tau)$. Performing mass renormalization and taking the limit $\delta\tau \to 0$, we obtain the equation of motion for the electron:

$$m\ddot{z}^{\alpha} = eF^{\text{in}\alpha}{}_{\beta}\dot{z}^{\beta} + \tfrac{2}{3}e^{2}(\dddot{z}^{\alpha} - \ddot{z}^{2}\dot{z}^{\alpha}) - \tfrac{1}{3}e^{2}(R_{\beta}{}^{\alpha}\dot{z}^{\beta} + \dot{z}^{\alpha}R_{\beta\gamma}\dot{z}^{\beta}\dot{z}^{\gamma}) + e^{2}\dot{z}^{\beta}\int_{-\infty}^{\tau} f^{\alpha}{}_{\beta\gamma'}\dot{z}^{\gamma'}(\tau')\,\mathrm{d}\tau' \tag{2.23}$$

where as usual

$$m = m_0 + \lim_{\delta\tau\to 0}(e^2/2\delta\tau). \tag{2.24}$$

Equation (2.23) is, of course, the one that follows from the DeWitt and Brehme procedure (see Hobbs 1968, equation (5.28)). The absence of the terms with the Ricci tensor in the final equation quoted by DB is due, as Hobbs (1968) points out, to the fact that the left-hand side of equation (5.11) in DB would be evaluated at the retarded and advanced proper times.

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References

Barut A O 1974 *Phys. Rev.* D **10** 3335
Barut A O and Villarroel D 1975 *J. Phys.* A: *Math. Gen.* **8** 156
DeWitt B S and Brehme R W 1960 *Ann. Phys., NY* **9** 220
Dirac P A M 1938 *Proc. R. Soc.* A **167** 148
Hobbs J M 1968 *Ann. Phys., NY* **47** 141
Tabensky R and Villarroel D 1975 *J. Math. Phys.* to be published